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Dilatations Revisited

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Dilatation, i.e. scale, symmetry in the presence of the dilaton in Minkowski space is derived from diffeomorphism symmetry in curved spacetime, incorporating the volume-preserving diffeomorphisms. The conditions for scale invariance are derived and their relation to conformal invariance is examined. In the presence of the dilaton scale invariance automatically guarantees conformal invariance due to diffeomorphism symmetry. Low energy scale-invariant phenomenological Lagrangians are derived in terms of dilaton-dressed fields, which are identified as the fields satisfying the usual scaling properties. The notion of spontaneous scale symmetry breaking is defined in the presence of the dilaton. In this context, possible phenomenological implications are advocated and by computing the dilaton mass the idea of PCDC (partially conserved dilatation current) is further explored.

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1. Introduction

Since the MIT-SLAC deep inelastic scattering experiments[1] the role of dilatations in physics has attracted fair amount of attention. Knowing the fact that we live in the world of a given scale, the classical scale symmetry[2][3][4][5] based on the dilatation of local coordinates in a Lorentz frame is destined to be broken at that given scale. Scale symmetry breaking in principle can be either explicit or spontaneous. It turns out that in a simple model with a scalar field the classical scale symmetry is anomalous at the quantum level, hence it is explicitly broken[4]. In more realistic cases like massless QED or gauge theories, scale symmetry is also broken by the trace anomaly[6]. Nevertheless, there have been attempts to introduce spontaneous breaking of scale symmetry with a Goldstone boson called the *dilaton*, as an analog to the pion for chiral symmetry breaking[7]. In either cases, no significant physical implications have been understood except for conformal field theories in two dimensions[8].

To further understand the role of scale symmetry in nature, we need to address the origin of the low energy scale symmetry. It is commonly believed that the scale symmetry in Minkowski space is an analog to the Weyl symmetry in curved spacetime. This however is in some sense unsatisfactory because of the lack of any direct relationship. A low energy scale transformation involves a change of local coordinates, but a Weyl transformation does not. We need a more direct connection between properties in curved spacetime and those in a local Lorentz frame particularly for any spacetime symmetries. This becomes an important issue if gravitational effects get stronger. Particularly, in string-motivated supersymmetric models the dynamics of the dilaton is crucial to understand the structure of the coupling constants and supersymmetry[9][10][11][12].

In this paper we start only with Diff (diffeomorphism) symmetry without referring to Weyl symmetry, then we shall derive the scale symmetry in Minkowski space¹. Note that in any dimensional spacetime scale invariance does not necessarily imply Weyl invariance, although it often implies conformal invariance. This signals that it would be better to understand scale symmetry as part of Diff, and that it could provide a natural explanation of the relation between

¹In [13] the relation between the low energy scale symmetry and the Weyl symmetry in curved spacetime is investigated and the role of restricted coordinate transformations in this context is suggested. They have tried to address the relevance of SDiff symmetry with respect to the dilaton, but do not succeed to address subtleties involving SDiff in curved spacetime, due to using coordinate dependent conditions. The covariant way to introduce SDiff in curved spacetime is explained in detail by the author in [14]. Nevertheless, there are certain similarities between some results in [13] and those of this paper, but careful readers will find the fundamentals are quite different.

scale symmetry and conformal invariance. Diff decomposes into SDiff (volume-preserving diffeomorphisms) and CDiff (conformal diffeomorphisms). Since SDiff preserves a volume element, dilatations are not part of SDiff. This is the crucial structure to be used in this paper.

Dilatations are defined in terms of local coordinates by

$$x \rightarrow e^\alpha x, \quad (1.1)$$

$$\Phi_{[d]}(x) \rightarrow e^{d\alpha} \Phi_{[d]}(e^\alpha x), \quad (1.2)$$

where d is the scale dimension (or the conformal weight). Eq.(1.1) suggests dilatations should be expressed as diffeomorphisms, although eq.(1.2) is not a result of a diffeomorphism. We however shall find that $\Phi_{[d]}$ can be expressed as a dilaton-dressed field and that eq.(1.2) indeed becomes a result of a diffeomorphism. This can be done only in the presence of the dilaton so that in this context scale symmetry naturally incorporates the dilaton. As an important result, scale invariance automatically guarantees conformal invariance because both are just part of Diff invariance.

Once quantum effects are included, subtraction of ultraviolet divergences inevitably demands the introduction of a renormalization scale as an explicit mass scale. This input scale is a free parameter and a theory should not depend on any changes of such a scale, yet it breaks scale symmetry in a naive sense. We find that at the quantum level, in the presence of the dilaton, the notion of the classical scale symmetry should be generalized to include changes of the renormalization scale. Then we obtain a conserved generalized scale current which incorporates naive scale anomalies. There is another anomaly if this quantum conservation law is not satisfied. As we shall find out, this in fact is consistent with the idea of the partially conserved dilatation current (PCDC), whilst the naive scale current is not. This quantum scale current conservation law produces an analog to the Callan-Symanzik equation of the effective potential without explicit variations of coupling constants. Then, we can define the spontaneous breaking of scale symmetry with the dilaton as a Goldstone boson, and that in a new symmetry-breaking vacuum, the PCDC structure is correctly produced with an anomalous term proportional to square of the dilaton mass. Without this modification, the leading term of the usual anomaly is not related to the dilaton. Therefore, it legitimizes our generalization.

In four dimensions, spontaneous symmetry breaking can be induced radiatively without using anomalies in this sense. It enables us to compute the dilaton mass explicitly, hence making the idea of PCDC realistic. Depending on the dilaton scale, various physics can be suggested. For example, if the dilaton scale is low, the dilaton can be light enough to be a

candidate for dark matter. Since the dilaton does not couple to gauge fields directly, but quite universally couples to fermions and scalars, the existence could be abundant, yet it could have escaped detections.

There is another implication. The naturalness of mass scales combined with the dilaton can also explain certain hierarchies of apparently different mass scales, with plausible assumptions, because dilaton contribution is often exponential.

This paper is organized as follows. In section two, dilatations are given in terms of the Diff in the geometry of $g_{\mu\nu} = e^{2\kappa\phi}\eta_{\mu\nu}$ incorporating the SDiff. Compared to Diff, Weyl transformations are not accompanied by changes of local coordinates. Then we recover the low energy dilaton transformation property. Low energy fields are in fact dilaton-dressed fields satisfying proper scaling properties, yet they have correct Lorentz properties. In this context, we can easily obtain the dilatation current and the conformal current to check their relationship explicitly. We recover the previously known results and the generalization in the presence of the dilaton. In section three, scale-invariant phenomenological Lagrangians are derived for various fields including the dilaton and the axion. In section four, using these phenomenological Lagrangians, we investigate plausible scenarios of scale symmetry breaking. In particular, introducing the notion of the spontaneous breaking of scale symmetry based on the generalized scale symmetry in the presence of the renormalization scale, we could further clarify the idea of PCDC. Also a possibility of explaining certain (mass) scale hierarchies using the dilaton is presented. Finally, in the last section, we summarize the results obtained and some possible future developments are proposed.

2. Defining Dilatations

2.1. Diff vs. Dilatations

In curved spacetime, a Weyl transformation is

$$\delta g_{\mu\nu} = 2\epsilon\rho(x)g_{\mu\nu}, \quad (2.1)$$

where $\rho(x)$ is constant for a global (or rigid) Weyl transformation. Usually in literatures this global Weyl transformation is regarded as the analog to a scale transformation in Minkowski space, hence relating scale symmetry to Weyl symmetry. But the awkwardness of this relation is that it does not naturally lead to the scale symmetry in Minkowski space by simply taking

the flat space limit of curved spacetime. Weyl transformations are supposed to be independent from coordinate changes contrary to scale transformations.

Our motivation is to introduce the scale symmetry in Minkowski space that can be naturally derived by simply taking the flat space limit of curved spacetime. Then it enables us to understand the origin of scale symmetry as part of spacetime symmetries. Furthermore, as soon to be explained, this naturally introduces the dilaton in Minkowski space and one can investigate scale-invariant Lagrangians in this context.

Under Diff, fields transform according to

$$(T_{\mu_1 \dots \mu_p} + \delta T_{\mu_1 \dots \mu_p}) dx^{\mu_1} \dots dx^{\mu_p} = T_{\mu_1 \dots \mu_p}(x + \delta x) d(x + \delta x)^{\mu_1} \dots d(x + \delta x)^{\mu_p}. \quad (2.2)$$

Then $\delta T_{\mu_1 \mu_2 \dots \mu_p}$ is nothing but the Lie derivative along δx . Now consider a metric of the form

$$g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}, \quad (2.3)$$

where $e^{n\kappa\phi} = \sqrt{g}$ in terms of $g \equiv |\det g_{\mu\nu}|$ and κ is the dilaton scale. The effect of introducing the explicit scale parameter, κ , is to let the dilaton have mass dimension $(n-2)/2$, where n is the dimension of spacetime. Note that κ is not really a free parameter because we can always rescale it by rescaling ϕ . As far as gravity is concerned, the natural choice of this scale is the Planck scale. But, here, instead of doing that, we will fix it later at any phenomenologically proper scale so that we can study the dilaton in an energy scale much lower than the quantum gravity scale. Since κ always appears in combination with ϕ , fixing κ actually requires a nontrivial dilaton vacuum expectation value.

As emphasized in ref.[14], if ϕ does not transform like a scalar under Diff, but the transformation property under Diff is dictated by that of the metric, then for $v \equiv \delta x$ in n dimensions $\delta g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ leads to

$$\delta e^{\kappa\phi} = \frac{1}{n} e^{\kappa\phi} D_\mu v^\mu, \quad (2.4)$$

where

$$D_\mu \equiv \partial_\mu + n\kappa \partial_\mu \phi. \quad (2.5)$$

In terms of g , $D_\mu \equiv \partial_\mu + \partial_\mu \ln \sqrt{g}$ and $\partial_\mu (\sqrt{g} v^\mu) = \sqrt{g} D_\mu v^\mu$. D_μ is the same as the covariant derivative ∇_μ only when it acts on a covariant vector, but, in general, they are different. Eq.(2.4) shows that eq.(2.3) is not to be considered as a conformal gauge fixing condition globally, but is a local expression of a metric in terms of a non-global function ϕ . For example, a density is not globally defined because it depends on a choice of local coordinates. If eq.(2.3) were the

conformal gauge fixing condition, it would lead to $\delta e^{2\kappa\phi} = v^\mu \partial_\mu e^{2\kappa\phi}$. Under SDiff, ϕ behaves like a constant to make pure ϕ Lagrangians manifestly SDiff-invariant.

The consistency condition between eq.(2.4) and eq.(2.3) is

$$\frac{2}{n} \eta_{\mu\nu} \partial_\alpha v^\alpha = \eta_{\mu\alpha} \partial_\nu v^\alpha + \eta_{\alpha\nu} \partial_\mu v^\alpha \quad (2.6)$$

so that diffeomorphisms of eq.(2.3) appear as conformal transformations of flat spacetime.

In particular, for infinitesimal α under the dilatation, eq.(1.1),

$$\delta\phi = \alpha \left(\frac{1}{\kappa} + x^\mu \partial_\mu \phi \right), \quad (2.7)$$

which is nothing but the dilatation property given in ref.[7]. This shows that the dilatations of the dilaton are results of diffeomorphisms.

Sometimes, it is useful to introduce a field redefinition (without the axion, see eq.(3.7) for the definition with the axion,)

$$\chi \equiv e^{\kappa\phi}. \quad (2.8)$$

Under Diff, χ transforms in a not-so-inspiring way, but, under dilatations

$$\delta\chi = \alpha (1 + x^\mu \partial_\mu) \chi. \quad (2.9)$$

Thus, although χ is not a scalar, it transforms like a scale-dimension-one field. χ is mass-dimensionless.

To produce eq.(1.2) let us introduce a dilaton-dressed field $\Phi_{[d]}$ as

$$\Phi_{[d]} \equiv e^{d\kappa\phi} \Phi, \quad (2.10)$$

where Φ transforms like a scalar under Diff. This dilaton dressing does not change the mass dimension of the field. Then under dilatations

$$\delta\Phi_{[d]} = \alpha(d + x^\mu \partial_\mu) \Phi_{[d]}. \quad (2.11)$$

Such dressing is not needed for vector fields in four dimensions because under Diff

$$\delta A_\mu = \alpha(1 + x^\lambda \partial_\lambda) A_\mu. \quad (2.12)$$

Similarly, we can define all dimensional fields in n dimensions by properly dressing with the dilaton and the scale transformation properties follow from the Diff transformation rule. In

this sense, the mass dimension of a field is not necessarily the same as the scale dimension. For example, the dilaton has mass dimension $(n-2)/2$, but its scale dimension is not even defined.

One can also easily check that the dilaton is, after all, a Lorentz scalar, hence so is $\Phi_{[d]}$. Thus, from the low energy point of view $\Phi_{[d]}$ and ϕ are indistinguishable from the usual scalar field. This clearly shows that the dilatations in Minkowski space can be derived from the Diff of virtual spacetime geometry of $g_{\mu\nu} = e^{2\kappa\phi}\eta_{\mu\nu}$ and we are never required to introduce Weyl symmetry.

2.2. Dilatation Current

The conformal current (of a second order system) can be easily computed as a Noether current of CDiff such that

$$\hat{J}_C^\mu = v^\lambda \tilde{T}_\lambda^\mu + \nabla_\lambda v^\lambda \tilde{K}^\mu + \nabla_\alpha \nabla_\lambda v^\lambda \hat{L}^{\alpha\mu}, \quad (2.13)$$

where $\nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2}{n} g_{\mu\nu} \nabla_\alpha v^\alpha$ and \tilde{T}_λ^μ , \tilde{K}^μ , $\hat{L}^{\alpha\mu} = \hat{L}^{\mu\alpha}$ are accordingly computed. The conformal invariance requires $\nabla_\mu \hat{J}_C^\mu = 0$, which must be satisfied if the Lagrangian is Diff invariant. For $g_{\mu\nu} = e^{2\kappa\phi}\eta_{\mu\nu}$ and a generic field Φ ,

$$\begin{aligned} \tilde{T}_\lambda^\mu &= e^{-n\kappa\phi} \left(-\delta^\mu_\lambda \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\lambda \Phi \right) \\ \tilde{K}^\mu &= \frac{1}{n\kappa} e^{-n\kappa\phi} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \partial_\mu \phi)} \right) \\ \hat{L}^{\alpha\mu} &= \frac{1}{n\kappa} e^{-n\kappa\phi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\alpha \phi)}. \end{aligned} \quad (2.14)$$

In the presence of the dilaton, the conformal current we have obtained contains an extra term compared to the one in [2] and [15], which is found by trial and error and also without the dilaton. Here, we have derived it as a Noether current with respect to the CDiff so that all the terms can be computed explicitly once an explicit Lagrangian is given.

To show the extra term explicitly, let us expand the covariant derivative in terms of the partial derivative to obtain

$$\hat{J}_C^\mu = v^\lambda \hat{T}_\lambda^\mu + \partial_\lambda v^\lambda \tilde{K}^\mu + \partial_\alpha v^\lambda \left(n\kappa \partial_\lambda \phi \hat{L}^{\alpha\mu} \right) + \partial_\alpha \partial_\lambda v^\lambda \hat{L}^{\alpha\mu}, \quad (2.15)$$

where

$$\hat{T}_\lambda^\mu = \tilde{T}_\lambda^\mu + n\kappa \partial_\lambda \phi \tilde{K}^\mu + n\kappa \partial_\alpha \partial_\lambda \phi \hat{L}^{\alpha\mu}. \quad (2.16)$$

The $\partial_\alpha v^\lambda$ term is extra. As in the flat case, this term can be rewritten as the second term using the conformal Killing vector condition, eq.(2.6), only if $\partial_\lambda \phi \hat{L}^{\alpha\mu}$ is symmetric under exchanging α and λ indices.

In our derivation of the conformal current, we assume the presence of the second order derivative term of the dilaton, which in fact exists in the curvature term. If we assume there are no second order derivative terms of any fields, then $\hat{L}^{\alpha\mu}$ vanishes. However, we shall find it necessary to keep them to prove conformal invariance.

The dilatation (or scale) current \hat{S}^μ is \hat{J}_C^μ with $v^\lambda = x^\lambda$:

$$\hat{S}^\mu = x^\lambda \hat{T}^\mu_\lambda + \hat{K}^\mu, \quad (2.17)$$

where

$$\hat{K}^\mu = n \widetilde{K}^\mu + n \kappa \partial_\lambda \phi \hat{L}^{\lambda\mu}. \quad (2.18)$$

In our case the dilatation current should be covariantly conserved with respect to the metric $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$. Thus

$$0 = \nabla_\mu \hat{S}^\mu = x^\lambda D_\mu \hat{T}^\mu_\lambda + \hat{T}^\mu_\mu + \nabla_\mu \hat{K}^\mu, \quad (2.19)$$

where D_μ is introduced in eq.(2.5). This can be satisfied if the stress-energy tensor is conserved as

$$D_\mu \hat{T}^\mu_\lambda = 0 \quad (2.20)$$

and

$$\hat{T}^\mu_\mu = -D_\mu \hat{K}^\mu. \quad (2.21)$$

Note that $\nabla_\mu \hat{T}^\mu_\lambda \neq 0$. In fact, we should not expect that $\nabla_\mu \hat{T}^\mu_\lambda = 0$ because \hat{T}^μ_λ is derived by choosing a local frame to fix coordinates such that $v^\lambda = x^\lambda$. On the contrary, the dilatation current \hat{S}^μ itself is generic so that it has to be covariantly conserved and its conservation is dictated by the Noether's theorem.

Using the property $\sqrt{g} D_\mu \hat{S}^\mu = \partial_\mu S^\mu$, where $S^\mu \equiv \sqrt{g} \hat{S}^\mu$, the above conditions can be written as the usual formulas in flat space. All other tensors relevant in flat space can be similarly defined by multiplying \sqrt{g} .

2.3. Conformal Invariance vs. Scale Invariance

In our case, Diff invariance leads to conformal invariance automatically. More precisely, conformal invariance requires that the conformal current \hat{J}_C^μ , eq.(2.13), should be covariantly

conserved such that

$$0 = \nabla_\mu \hat{J}_C^\mu = v^\lambda \nabla_\mu \tilde{T}_\lambda^\mu + \nabla_\alpha v^\alpha \left(\frac{1}{2} \tilde{T}_\mu^\mu + \nabla_\mu \tilde{K}^\mu \right) + \nabla_\mu \nabla_\alpha v^\alpha \left(\tilde{K}^\mu + \nabla_\lambda \hat{L}^{\mu\lambda} \right) + \nabla_\mu \nabla_\lambda \nabla_\alpha v^\alpha \hat{L}^{\mu\lambda}. \quad (2.22)$$

In general, the terms in the RHS are not independent so that conformal invariance does not necessarily require each term to vanish separately in curved spacetime. In a generic second order system, $\hat{L}^{\mu\lambda} = g^{\mu\lambda} \hat{L}/n$. For this reason, one might be tempted to demand $\nabla_\alpha v^\alpha$ is harmonic and each term vanishes separately. But, this does not happen. One obvious reason is that \tilde{T}_λ^μ is not necessarily a covariantly conserved stress-energy tensor in curved spacetime.

For our purpose, as long as the conformal current is covariantly conserved, we do not need covariant conditions on other terms because we are interested in conditions to be imposed in a local Lorentz frame. Nevertheless, since any metric can be written as $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$ for nonscalar ϕ , we can still retain all necessary geometric data[14].

In a local Lorentz frame we can choose coordinates such that v^λ , $\partial_\alpha v^\lambda$, $\partial_\alpha \partial_\beta v^\lambda$...etc., are linearly independent. Then

$$\begin{aligned} 0 = \nabla_\mu \hat{J}_C^\mu &= v^\lambda D_\mu \hat{T}_\lambda^\mu \\ &+ \partial_\mu v^\lambda \left(\hat{T}_\lambda^\mu + D_\alpha \left(\delta_\lambda^\mu \tilde{K}^\alpha + \partial_\lambda \ln \sqrt{g} \hat{L}^{\alpha\mu} \right) \right) \\ &+ \partial_\mu \partial_\sigma v^\lambda \left(\delta_\lambda^\sigma \left(\tilde{K}^\mu + D_\alpha \hat{L}^{\alpha\mu} \right) + \partial_\lambda \ln \sqrt{g} \hat{L}^{\sigma\mu} \right) \\ &+ \partial_\mu \partial_\lambda \partial_\alpha v^\alpha \hat{L}^{\sigma\mu} \end{aligned} \quad (2.23)$$

implies the four terms in the RHS vanish independently, leading to

$$0 = D_\mu \hat{T}_\lambda^\mu, \quad (2.24)$$

$$0 = \partial_\mu v^\lambda \left(\hat{T}_\lambda^\mu + D_\alpha \left(\delta_\lambda^\mu \tilde{K}^\alpha + \partial_\lambda \ln \sqrt{g} \hat{L}^{\alpha\mu} \right) \right), \quad (2.25)$$

$$0 = \partial_\mu \partial_\sigma v^\lambda \left(\delta_\lambda^\sigma \left(\tilde{K}^\mu + D_\alpha \hat{L}^{\alpha\mu} \right) + \partial_\lambda \ln \sqrt{g} \hat{L}^{\sigma\mu} \right), \quad (2.26)$$

$$0 = \partial_\mu \partial_\lambda \partial_\alpha v^\alpha \hat{L}^{\sigma\mu}. \quad (2.27)$$

Eq.(2.6) implies that eq.(2.27) is true if $\hat{L}^{\mu\lambda} = g^{\mu\lambda} \hat{L}/n$ for $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$. Note that in eqs.(2.25)(2.26) we require the whole thing to vanish instead of the formulae inside the bracket. It is because these conditions in fact depend on the specific details of v^λ . This is one of the reasons the conformal structure actually depends on the dimensionality of spacetime.

In Euclidean two dimensions, the second term in eq.(2.26) also vanishes because v^λ itself is harmonic. Then, using the trace condition eq.(2.21), one can show that eq.(2.25) is satisfied.

The only remaining condition is

$$0 = \widetilde{K}^\mu + D_\alpha \widehat{L}^{\alpha\mu}. \quad (2.28)$$

In Minkowski two dimensions, although we cannot use the power of complex analysis, yet the Diff invariance dictates v^λ should be still harmonic. In fact, as far as physics is concerned, we can impose the light-front conditions to derive equivalent conditions. Therefore, in two dimensions for scale symmetry to imply conformal invariance, one extra condition eq.(2.28) needs to be satisfied. Expressing in the flat space objects, this is the same condition as the one given by Polchinski[15].

In other n dimensions, one can use explicit special conformal transformations

$$v^\lambda = a_\alpha (-\eta^{\alpha\lambda} x^2 + 2x^\alpha x^\lambda). \quad (2.29)$$

Regardlessly of any $\widehat{L}^{\sigma\mu}$, eq.(2.27) is true and eq.(2.25) generally follows from scale invariance. The extra condition, eq.(2.26), now reads

$$n (\widetilde{K}^\mu + D_\alpha \widehat{L}^{\mu\alpha}) + 2\partial_\alpha \ln \sqrt{g} \widehat{L}^{\mu\alpha} - \eta^{\mu\nu} \partial_\nu \ln \sqrt{g} \eta_{\lambda\alpha} \widehat{L}^{\lambda\alpha} = 0. \quad (2.30)$$

Due to the dilaton the extra terms compared to eq.(2.28) do not vanish. For $\widehat{L}^{\mu\lambda} = g^{\mu\lambda} \widehat{L}/n$ the extra terms contain a prefactor $(2 - n)$ so that they vanish only in two dimensions. Note that, as long as the action is Diff invariant, eq.(2.28) and eq.(2.30) are supposed to be satisfied automatically. This can be achieved because of the dilaton.

This in turn implies that scale symmetry breaking amounts Diff symmetry breaking of $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$ and remaining symmetry is supposed to be SDiff symmetry, although only Poincaré symmetry is usually manifest in flat spacetime due to a gauge choice.

3. Scale-invariant Phenomenological Lagrangians

Scale-invariant Lagrangians are nothing but the usual Lagrangians of tensor fields defined in the curved spacetime of the dilaton virtual geometry. Then we simply express them in terms of the dilaton-dressed fields.

3.1. Dilaton

The kinetic energy for the dilaton in Minkowski space can be derived from the curvature of the virtual geometry of metric $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$. Let us first consider the stringy Lagrangian

$$\mathcal{L}_S = -\frac{1}{2\kappa^2} \sqrt{g} e^{-2\Phi} (R - 4g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi).$$

One can quickly check that $\phi \propto \Phi$ does not lead to a scale invariant action for the above metric. Although it is possible[16], here we do not want to have another gravitational scalar field, so we shall seek a case without one. On the contrary, the only scale invariant Lagrangian we can obtain is to let $g_{\mu\nu} = \eta_{\mu\nu}$ and then $\Phi = -\frac{n}{2}\phi$. In other words, in the string frame once the dilaton is identified separately[17], the metric should not contain any dilaton degrees of freedom so that the metric should be taken to be flat for our purpose. Thus the stringy dilaton and the low energy dilaton become equivalent only if we incorporate SDiff. Anyhow, this is equivalent to choosing the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{EH}} = -\sqrt{g}R/\mathcal{N}\kappa^2 \quad (3.1)$$

where \mathcal{N} is a normalization numerical prefactor. At the scale in which the gravity is relevant, one should choose $\mathcal{N} = 2$ and $\kappa^2 = 8\pi G$. But, here we shall choose them to make the kinetic energy term canonical. It also confirms that in our case scale symmetry is not related to the global Weyl symmetry because \mathcal{L}_{EH} is not Weyl invariant if $n \neq 2$.

For the given metric in n dimensions ($n \neq 2$)

$$\mathcal{L}_{\text{EH}} = -\frac{1}{\mathcal{N}}(n-1)(n-2)e^{(n-2)\kappa\phi}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \text{total derivative} \quad (3.2)$$

In case of proving conformal invariance the total derivative term is important. Note that it has an analogous form to the nonlinear chiral Lagrangian except the prefactor of ϕ is real.

In two dimensions \mathcal{L}_{EH} does not provide the dilaton kinetic energy. Instead, we could use

$$\mathcal{L}_{\text{L}} = -\frac{1}{8}\sqrt{g}R\Delta^{-1}R = \frac{1}{2}\kappa^2\eta^{\mu\nu}\phi\partial_\mu\partial_\nu\phi, \quad (3.3)$$

which is nothing but the Liouville Lagrangian. In two dimensions it is more natural to absorb κ into ϕ so that ϕ becomes mass-dimensionless.

The main difficulty of understanding the dilaton dynamics lies on the undesirable structure of its potential energy. From the curved spacetime point of view the only allowed tree level potential is the exponential one, hence there is no stable dilaton vacuum. The well known difficulty of handling such a case is typified in the Liouville theory in two dimensions[18][19][14]. This is also partly a source of the “runaway dilaton problem” in supergravity models[10]. Without such a tree level potential, the dilaton generates a so-called flat direction along which vacua are degenerate.

As alluded in refs.[14][16], we expect that this difficulty might be overcome if we abandon the manifest Diff invariance and rely on SDiff invariance only. Without Diff, at least the symmetry

alone does not forbid other stabilizing potentials. In fact careful observation reveals that the dilaton ϕ is not even an ordinary scalar field as shown in eq.(2.4). In our case, being in a local Lorentz frame, we can accommodate this structure naturally for the low energy dilaton. Then, in the next section, we shall derive the effective potential that indeed has a symmetry breaking vacuum.

3.2. Scalar Fields

The scale invariant lagrangian for a scalar field is

$$\mathcal{L}_\Phi = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu - d\kappa\partial_\mu\phi)\Phi_{[d]}(\partial_\nu - d\kappa\partial_\nu\phi)\Phi_{[d]} - \frac{1}{2}m^2e^{2\kappa\phi}\Phi_{[d]}^2 - \dots, \quad (3.4)$$

where $\Phi_{[d]}$ the dilaton-dressed scalar field (eq.(2.10)) and $d \equiv (n-2)/2$, which is nothing but the mass dimension of the dilaton². $\kappa\partial_\mu\phi$ acts like minimal coupling of a gauge field with a charge id and the dilaton-scalar couplings are derivative. In two dimensions, in particular, this dilaton coupling disappears so that the scalar field does not interact with the dilaton. In other dimensions the dilaton-scalar couplings are nonrenormalizable, hence the Lagrangian appears as an effective field theory Lagrangian[20].

3.3. Fermions

Similarly, in terms of dilaton-dressed fermionic fields $\psi_{[d']} \equiv e^{d'\kappa\phi}\psi$ with $d' \equiv (n-1)/2$,

$$\mathcal{L}_\psi = i\bar{\psi}_{[d']}\gamma^\mu(\partial_\mu - d'\kappa\partial_\mu\phi)\psi_{[d']} - m'e^{\kappa\phi}\bar{\psi}_{[d']}\psi_{[d]}, \quad (3.5)$$

where the gamma matrices are those in Minkowski space. The dilaton-fermion coupling is also derivative and non-renormalizable. The dilaton interacts with fermions in any $d > 1$ dimensions.

3.4. Gauge Fields

We shall always choose gauge fields to be scale-dimension one so that they are not dressed by the dilaton. In this way, we can relate the dilaton with the gauge coupling constant.

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}e^{(n-4)\kappa\phi}\eta^{\mu\lambda}\eta^{\nu\sigma}\text{Tr}F_{\mu\lambda}F_{\nu\sigma}. \quad (3.6)$$

²This scale invariant Lagrangian is also derived in [13] and the relevance of the conformally flat metric, eq.(2.3), is noticed.

In four dimensions, the dilaton and gauge fields decouple. In other than four dimensions, the effective gauge coupling constant depends on the dilaton as in string motivated models.

3.5. Axion

As a matter of fact, there is another way to define χ than eq.(2.8). Simply allow χ to be complex such that $\chi^*\chi = e^{2\kappa\phi}$. Then, there is an undetermined phase a , which we shall call the axion, so that

$$\chi \equiv e^{\kappa(\phi+ia)}. \quad (3.7)$$

Under Diff, this axion transforms like a (pseudo-)scalar so that, in particular, under dilatations, χ still transforms according to eq.(2.9).

A real scalar field should be still dressed as eq.(2.10), but a charged complex field now can be dressed by the complex χ . This in fact is consistent with the low energy axion associated with $U(1)_{PQ}$ because the axion is only associated with a charged scalar. Despite this fact, note that the axion we defined here is not the low energy axion[21]. Being part of the geometrical data, it is rather the gravitational axion associated with the dilaton[9]. Nevertheless, upon dressing, this axion mixes with the low energy axion so that it might not be distinguishable in practice.

The metric being real, the kinetic term of the axion cannot be derived from the curvature term. Thus the axion kinetic term can be introduced by hand as

$$\mathcal{L}_a = -\frac{1}{\mathcal{N}}e^{(n-2)\kappa\phi}\eta^{\mu\nu}\partial_\mu a\partial_\nu a, \quad (3.8)$$

where $\mathcal{N} = \frac{2}{(n-2)^2}$ if $n \neq 2$ and $\mathcal{N} = 2$ if $n = 2$. Combined with \mathcal{L}_ϕ that can be read off from \mathcal{L}_{EH} or \mathcal{L}_L , we can obtain

$$\mathcal{L}_\chi \equiv \mathcal{L}_\phi + \mathcal{L}_a = -\frac{1}{2\kappa^2}\eta^{\mu\nu}\partial_\mu\chi_d^*\partial_\nu\chi_d, \quad (3.9)$$

where $\chi_d \equiv \chi^d$ for $d = (n-2)/2$ if $n \neq 2$ and $\chi_d \equiv \kappa(\phi+ia)$ for $d = 1$ if $n = 2$. In four dimensions the axion term can be naturally derived from antisymmetric field $H_{\mu\nu\lambda}$ such that $\partial_\sigma a = (1/6)\epsilon_{\sigma\mu\nu\lambda}H^{\mu\nu\lambda}$ [9].

In this case, the dilaton-dressed charged scalar field is

$$\Phi_{[d]} = \chi \Phi \quad (3.10)$$

so that the scale-invariant scalar Lagrangian coupled to gauge fields becomes

$$\mathcal{L}_\Phi = -\eta^{\mu\nu} \left(\mathcal{D}_\mu \Phi_{[d]} \right)^\dagger \mathcal{D}_\nu \Phi_{[d]}, \quad (3.11)$$

where $\mathcal{D}_\mu \equiv [\partial_\mu - igA_\mu - d\kappa\partial_\mu(\phi + ia)]$. Unlike the dilaton, in the presence of U(1) gauge field the axion can be gauged away if the global part of U(1) is of Peccei-Quinn type. This means that the presence of the axion breaks not only U(1)_{PQ} but also this U(1). To avoid breaking the local U(1) one can always insist the axion does not depend on the U(1) gauge degrees of freedom, but accidental alignment of the axion direction and U(1) gauge degrees may not be unavoidable. For other gauge fields axion couplings are given according to properly modified \mathcal{D}_μ . For example, for fermions in eq.(3.5) $(\partial_\mu - d'\kappa\partial_\mu\phi)$ should be replaced with \mathcal{D}_μ which axial axion coupling, which can be done by dressing Weyl fermions with the complex χ .

3.6. Example: Dilaton-Scalar System

Let us demonstrate the previous symmetries for the dilaton-scalar Lagrangian in four dimensions:

$$\mathcal{L} = \frac{1}{2\kappa} e^{2\kappa\phi} \eta^{\mu\nu} \partial_\mu \partial_\nu \phi + \frac{1}{2} e^{2\kappa\phi} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta^{\mu\nu} (\partial_\mu - \kappa \partial_\mu \phi) \Phi_{[1]} (\partial_\nu - \kappa \partial_\nu \phi) \Phi_{[1]}. \quad (3.12)$$

We explicitly restored the second order derivative term for conformal invariance. Actual computation is easier in terms of $\Phi = e^{-\kappa\phi} \Phi_{[1]}$, but the result is equivalent. The equations of motion are

$$0 = \eta^{\mu\nu} (\partial_\mu + 2\kappa \partial_\mu \phi) \partial_\nu \Phi \quad (3.13)$$

$$0 = \eta^{\mu\nu} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{\kappa} (\partial_\mu + \kappa \partial_\mu \phi) \partial_\nu \phi \right). \quad (3.14)$$

Relevant operators are

$$\begin{aligned} \widetilde{K}^\mu &= 0, \quad \widehat{L}^{\mu\nu} = \frac{1}{8\kappa^2} e^{-2\kappa\phi} \eta^{\mu\nu} \\ \widehat{T}^\mu_\lambda &= e^{-2\kappa\phi} \eta^{\mu\nu} \left(\frac{1}{2\kappa} \partial_\nu \partial_\lambda \phi - \partial_\nu \Phi \partial_\lambda \Phi \right). \end{aligned} \quad (3.15)$$

Then

$$D_\mu \widehat{T}^\mu_\lambda = (\partial_\mu + 4\kappa \partial_\mu \phi) \widehat{T}^\mu_\lambda = 0, \quad (3.16)$$

$$\widehat{T}^\mu_\mu + D_\mu \widehat{K}^\mu = 0 \quad (3.17)$$

are easily satisfied so that \mathcal{L} is scale invariant. Using eq.(2.30), it can be shown that for the given $\widehat{L}^{\mu\nu}$ the extra condition for \mathcal{L} to be conformally invariant becomes simply $\widetilde{K}^\mu = 0$, which

is indeed satisfied in eq.(3.15). Without the second order derivative term in eq.(3.12), \mathcal{L} does not lead to a conserved conformal current. Despite that two Lagrangians determine the same scale-invariant classical theory if the difference is a divergence, only one leads to the correct conserved conformal current. This is because the divergence term is necessary for the dilaton to be Diff-invariant. It also indicates that equations of motion are more fundamental than a Lagrangian, as far as symmetries are concerned³. Total divergence terms in a Lagrangian will be automatically taken care of in the symmetry analysis of equations of motion.

4. Scale Symmetry Breaking

Scale symmetry is a continuous global spacetime symmetry based on rigid transformations. It is natural to expect that the symmetry will be broken to provide us a scale to live on. As we can see below, there are many different ways to break scale symmetry. The most common case is that scale symmetry is explicitly broken because it is anomalous due to quantum effects. This happens because of the appearance of the renormalization scale. Exceptional cases can occur only if beta-functions of coupling constants vanish. But in the presence of the dilaton, this is the least interesting case simply because it does not allow any particles associated with the symmetry breaking. Thus we shall not consider explicit breaking cases here. Having the dilaton means that scale symmetry is presumed to be spontaneously broken. From here on, we shall focus on four dimensional cases, unless specified otherwise.

4.1. Spontaneous Scale Symmetry Breaking

The scale invariant Lagrangians we derived in the previous section already define effective field theories because nonrenormalizable higher order dilaton terms are involved. The potential term is not yet introduced. One can explicitly introduce scale-violating dilaton potential to provide the dilaton mass[7], but here we would like to do without explicitly introducing symmetry breaking dilaton potentials. As we shall soon find out, the 1PI effective potential, generated from the typical scale-invariant tree level dilaton potential, induces spontaneous breaking of scale symmetry⁴. Despite that the symmetry in consideration is a continuous one, the dilaton becomes massive in the new vacuum. This is not really unusual because, unlike in other cases with continuous internal symmetries, there is only one real field is involved. So it imitates the

³For some description of symmetry structure of equations of motion in general and applications, see ref.[22].

⁴The subtleties involving the derivation of 1PI effective potential from an (Wilsonian) effective field theory, which is nonrenormalizable, will be addressed later.

case of spontaneous breaking of a discrete symmetry, which does not involve massless Goldstone bosons. Perhaps, this indicates the dilaton is in fact part of the graviton and scale symmetry breaking as part of Diff contains an analog to the Higgs mechanism to make the dilaton massive as advocated in ref.[16]. Thus the 1PI effective potential simply shows the result of this Higgs mechanism and the dilaton has eaten up an implicit Goldstone boson.

Before we begin the actual computation, let us first clarify what we mean by “spontaneous” scale symmetry breaking here, since we intend to derive this breaking radiatively. Any renormalization of an ultraviolet divergent process introduces a subtraction (or, renormalization) scale as an explicit mass scale, hence breaking scale symmetry in a naive sense. Unless the beta function vanishes, the effect of such breaking appears as an anomaly. As we mentioned before, if an anomaly breaks scale symmetry, the introduction of the dilaton is meaningless. Thus, in the presence of the dilaton, we need to carefully understand the meaning of the subtraction scale.

As we shall see, both 1PI effective potentials for the dilaton and a scalar, $V_{\phi,\text{eff}}$ and $V_{\Phi,\text{eff}}$ respectively, are scale invariant, if we allow the subtraction scale changes under infinitesimal scale transformations eq.(1.1) as

$$\delta M = \alpha M. \quad (4.1)$$

While $V_{\Phi,\text{eff}}$ still has a scale invariant invariant vacuum, $V_{\phi,\text{eff}}$ no longer has a scale invariant vacuum. Thus, including eq.(4.1) as part of the scale transformation makes the effective action preserve scale symmetry in the presence of the dilaton as it should be, yet there is a new vacuum in which this “generalized” scale symmetry is spontaneously broken. Furthermore, this is also perfectly consistent with the idea of PCDC as we shall show later.

Consider the pure dilaton Lagrangian in four dimensions given by

$$\mathcal{L}_\phi = -\frac{1}{2}e^{2\kappa\phi}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\Lambda}{4}e^{4\kappa\phi}. \quad (4.2)$$

This is a four dimensional analog to the two dimensional Liouville Lagrangian in the sense that eq.(3.2) with the cosmological constant term for $n = 2$ is the Liouville Lagrangian if $\mathcal{N} = 2(n-1)(n-2)$. (Or, more precisely, eq.(3.3).) The presence of the prefactor $e^{2\kappa\phi}$ can be absorbed into the derivative by $\chi \equiv e^{\kappa\phi}$ so that we have some resemblance to the nonlinear chiral Lagrangian of pions except χ here is real and contains only one real field.

To compute the effective potential, first of all, we need to locate a vacuum. Since the derivative terms do not contribute in determining a vacuum, \mathcal{L}_ϕ does not have a stable vacuum,

but only a runaway vacuum. In practice, we can still use the asymptotic vacuum, which is the limit of the runaway vacuum and technically stable, to compute the effective potential. This is good enough because the functional integral in fact only requires an asymptotic vacuum. Also, for an effective field theory, nonrenormalizable terms are irrelevant particularly in the low momentum regime, we can still compute the effective potential for renormalizable interactions. Even in the high momentum regime, if the effective field theory for the dilaton is derived from a well-defined renormalizable theory in the quantum gravity scale, we can still have plenty of counter terms to tame all the higher order terms in the functional integral. In this sense, the square term that is needed for the 1PI effective potential can be assumed to be fairly unambiguous.

Note that eq.(4.2) indeed has an asymptotic vacuum that is the limit of the long tail, since the potential is bounded below. So we compute the 1PI dilaton effective potential to obtain

$$V_{\phi,\text{eff}} = \frac{\Lambda}{4} e^{4\kappa\phi} + \frac{1}{64\pi^2} \left(4\kappa^2 \Lambda e^{2\kappa\phi} \right)^2 \left(\log \frac{4\kappa^2 \Lambda e^{2\kappa\phi}}{M^2} - \frac{3}{2} \right). \quad (4.3)$$

One can easily check that this effective potential is scale invariant incorporating eq.(4.1). Without eq.(4.1), the $\phi e^{4\kappa\phi}$ term is not scale invariant.

$V_{\phi,\text{eff}}$ has a new minimum located at

$$\langle \phi \rangle = \frac{1}{2\kappa} \left(1 - \frac{\pi^2}{\kappa^4 \Lambda} + \log \frac{M^2}{4\kappa^2 \Lambda} \right) \quad (4.4)$$

for any $\kappa^4 \Lambda$. And in this new vacuum the scale symmetry is spontaneously broken in the sense we explained before. In fact the derivative term in eq.(4.2) also breaks the generalized scale symmetry after shifting to the new vacuum. In the context of $g_{\mu\nu} = e^{2\kappa\phi} \eta_{\mu\nu}$, this corresponds to spontaneous symmetry breaking of Diff to SDiff, as advocated in ref.[16].

This spontaneous breaking of scale symmetry can also be elegantly described in terms of an analog to the Callan-Symanzik equation of the effective potential. Without introducing the renormalized coupling constant, under the generalized scale invariance the invariant effective potential satisfies

$$\left(M \frac{\partial}{\partial M} + \frac{1}{\kappa} \frac{\partial}{\partial \phi} + d\Phi_{[d]} \frac{\partial}{\partial \Phi_{[d]}} - n \right) V_{\text{eff}}(M, \phi, \Phi_{[d]}) = 0. \quad (4.5)$$

Note that an anomaly with respect to the usual scale symmetry corresponds to

$$\partial_\mu S^\mu = -M \frac{\partial V_{\text{eff}}}{\partial M}. \quad (4.6)$$

But this is absorbed into the equation and we would not count it as an anomaly under the generalized scale symmetry. After shifting the vacuum, when the equation is rewritten in terms of new variables, if there is a term violating this form of an equation, the scale symmetry is spontaneously broken. As one can easily check, if the dilaton effective potential has a new vacuum with a nontrivial vacuum expectation value, then the above equation is not satisfied after shifting the vacuum because of additional term $M(\partial\langle\phi\rangle/\partial M)(\partial V_{\text{eff}}/\partial\phi)$. This will appear as anomalous dilatation current conservation law

$$\partial_\mu S_Q(M, \phi) = -M \frac{\partial\langle\phi\rangle}{\partial M} \frac{\partial V_{\text{eff}}}{\partial\phi} \Big|_{\phi+\langle\phi\rangle}, \quad (4.7)$$

where S_Q is the generalized dilatation current.

4.2. Via Internal Symmetry Breaking: Without Dilaton Loops

We can also break scale symmetry in connection with the spontaneous symmetry breaking of internal symmetries. This can be done because the latter involves a vacuum expectation value of a field, which happens to fix a scale. Thus scale symmetry should be expected to be broken at the same time.

For the argument's sake, let us consider the dilaton-scalar system with \mathbb{Z}_2 symmetry and introduce internal-symmetry breaking term

$$V(\Phi_{[1]}) = \frac{\lambda}{4} \left(\Phi_{[1]}^2 - v^2 \right)^2. \quad (4.8)$$

This tree level scalar potential contains explicit scale symmetry breaking terms: λv^4 and $\lambda v^2 \Phi_{[1]}^2$. Thus, scale symmetry is explicitly broken by hand. Perturbed around a new vacuum, the potential yields terms that break both internal symmetry and scale symmetry.

We might ask what happens to the dilaton, since scale symmetry is broken. Note that the kinetic energy term of the scalar field leads to a term which modifies the dilaton kinetic energy term so that the dilaton Lagrangian, eq(4.2), now reads

$$\tilde{\mathcal{L}}_\phi = -\frac{1}{2} \left(e^{2\kappa\phi} + v^2 \kappa^2 \right) \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\Lambda}{4} e^{4\kappa\phi}. \quad (4.9)$$

By shifting ϕ , $v^2 \kappa^2$ term cannot be removed. Thus, the vacuum is still a runaway vacuum, despite the fact that scale symmetry is broken. This is not unexpected. Since scale symmetry is explicitly broken, there is no reason to introduce the dilaton from the beginning. As a matter of fact, near the runaway vacuum the dilaton decouples from the scalar system. The situation does not become better for field redefinition $\chi \equiv e^{\kappa\phi}$ to allow χ to shift instead of ϕ .

For a continuous internal symmetry, say, $U(1)$, we can introduce the axion according to the previous section. Due to the axion-scalar coupling, upon breaking of $U(1)$, the axion kinetic energy can be modified too. For the dilaton-axion kinetic energy, eq.(3.9), any modification of the prefactor of the dilaton kinetic energy must be accompanied by that of the axion. And the prefactor cannot be removed by shifting χ . Thus, it is not necessary for the dilaton to obtain a nontrivial vacuum expectation value if scale symmetry breaking is due to an explicit scale symmetry breaking term in a Lagrangian.

Let us next check a case without putting an explicit symmetry breaking term. For a tree level dilaton-scalar coupling without derivatives, we can introduce a simple scale invariant term

$$-\frac{1}{2}m^2 e^{2\kappa\phi} \Phi_{[1]}^2. \quad (4.10)$$

Combined with the tree level dilaton potential, the potential in this case reads

$$V(\phi, \Phi) = \frac{\lambda}{4} \left(\Phi_{[1]}^2 - v^2 e^{2\kappa\phi} \right)^2 + \frac{1}{4} \left(\Lambda - \lambda v^4 \right) e^{4\kappa\phi}. \quad (4.11)$$

The vacuum conditions are $\Phi_{[1]}^2 = v^2 e^{2\kappa\phi}$ and $e^{2\kappa\phi} = 0$, hence there is no spontaneous breaking of internal symmetry. This symmetry breaking can occur only if the dilaton gets a nontrivial vacuum expectation value, that is, scale symmetry is also broken. Therefore, we need to introduce explicit scale symmetry breaking term for tree level symmetry breaking. Furthermore, to break scale symmetry at the same time to obtain a nontrivial dilaton vacuum expectation value we need polynomial terms of ϕ .

The radiative case is slightly different. Let us consider scale-invariant scalar self-interaction $\lambda \Phi_{[1]}^4$ without a mass term. In four dimensions, no explicit dilaton is involved in this interaction. Then one-loop contributions of the scalar field generate a double well effective potential with a new vacuum[23]. Perturbed around this new vacuum, $\Phi_{[1]} = v + \varphi$, the effective potential now contains a term $\lambda v^2 \varphi^2$. The presence of such a term breaks the classical scale symmetry, which is the case with $\delta M = 0$. In fact, the classical scale symmetry is broken by any term other than $\Phi_{[1]}^4$. Under the generalized scale symmetry, one can easily check that the effective potential is invariant both before and after shifting the $\Phi_{[1]}$ vacuum.

4.3. Simultaneous Symmetry Breaking: The Internal and Scale

In the previous section, we noticed that internal symmetry breaking in the scalar sector alone does not generate a nontrivial dilaton potential to fix the dilaton vacuum expectation

value, despite that apparent scale symmetry breaking occurs. This inconsistency simply tells us that such a semiclassical way of deriving internal symmetry breaking in the presence of the dilaton is not correct. The dilaton should be accounted in the same way.

For this purpose, we compute the full effective action

$$V_{\text{eff}}(\phi, \Phi) = V_{\phi, \text{eff}} + V_{\Phi, \text{eff}}, \quad (4.12)$$

where, in the one scalar case, $V_{\Phi, \text{eff}}$ is the usual Coleman-Weinberg effective potential

$$V_{\Phi, \text{eff}} = \frac{\lambda}{4} \Phi_{[1]}^4 + \frac{1}{64\pi^2} \left(3\lambda \Phi_{[1]}^2 \right)^2 \left(\log \frac{3\lambda \Phi_{[1]}^2}{M^2} - \frac{3}{2} \right) \quad (4.13)$$

and $V_{\phi, \text{eff}}$ is given in eq.(4.3). There is no ϕ - Φ mixed term in the effective potential because dilaton-scalar couplings in the tree Lagrangian are all derivative ones.

In this case, the vacuum expectation values of the scalar and the dilaton are related in terms of the common renormalization scale M so that

$$\langle \phi \rangle = \frac{1}{2\kappa} \left(\frac{16\pi^2}{9\lambda} - \frac{\pi^2}{\kappa^4 \Lambda} + \log \frac{3\lambda \langle \Phi_{[1]} \rangle^2}{4\kappa^2 \Lambda} \right). \quad (4.14)$$

Thus, scale symmetry as well as internal symmetry are broken at the same time.

The naturalness, which means that all the dimensionful parameters in a theory should be of the same order, implies $\Lambda^{1/4} \sim \langle \Phi_{[1]} \rangle \sim \langle \phi \rangle$. This in turn implies

$$\kappa^4 \Lambda \sim \kappa^2 \langle \Phi_{[1]} \rangle^2 \sim \kappa \langle \phi \rangle. \quad (4.15)$$

For eq.(4.14), the above is consistent with

$$\langle \phi \rangle \sim \mathcal{O}(1/\kappa). \quad (4.16)$$

Thus in the new vacuum the dilaton scale κ determines the scale of the theory.

Of course, though less natural, a theory can have different mass scales. Likewise, the internal symmetry breaking scale and the scale symmetry breaking scale can be different. But, these scales must be built in somehow.

4.4. Effects of Scale Symmetry Breaking on Mass Scales

The naturalness of mass scales in a theory has another interesting implication. Consider spontaneous symmetry breaking involving two different vacuum expectations, which is a common situation in electroweak symmetry breaking in supersymmetric models[24][25][26]. There

is an undetermined parameter $\tan \beta = v_2/v_1$, where v_1 and v_2 are vacuum expectation values for two different scalars. It is very unnatural to have two completely different dimensionful parameters[27]. In the presence of the dilaton, this unnaturalness can be accommodated even if it ever happens.

Let one of the scalar field have a scale-invariant mass term so that $\lambda_1 \left(|\Phi_1|^2 - v^2 e^{2\kappa\phi} \right)^2$ for some reason and the other have a scale-breaking mass term so that $\lambda_2 \left(|\Phi_2|^2 - v^2 \right)^2$. Then symmetry breaking leads to $v_1 = v$ and $v_2 = v e^{\kappa\langle\phi\rangle}$. This leads to

$$\tan \beta = e^{\kappa\langle\phi\rangle}.$$

Thus we only need one mass scale v . Note that, being exponential, $\tan \beta$ can be fairly large even for $\kappa\langle\phi\rangle \sim \mathcal{O}(1)$.

Since we do not like to have an explicit scale-breaking term, in principle the above could be more realistically checked in the radiative case. In principle, the parameters to determine the two vacuum expectation values are coupling constants, the renormalization scale and the dilaton scale. Upon scale symmetry breaking, the dilaton vacuum expectation value can eliminate the renormalization scale. Therefore, the two vacuum expectation values should be related by the dilaton one, which always enters exponentially.

4.5. PCDC

$\mathcal{L}_{\text{eff}}(\phi)$ in the new vacuum is a good candidate to address the PCDC. After shifting the dilaton vacuum, the effective potential reads

$$V_{\phi,\text{eff}} = \exp \left\{ 2 \left(1 - \frac{\pi^2}{\kappa^4 \Lambda} \right) \right\} \frac{M^4}{128\pi^2} e^{4\kappa\phi} (4\kappa\phi - 1). \quad (4.17)$$

Let us first check the case of the classical scale symmetry $\delta M = 0$. Under the variation, the leading scale symmetry breaking term is a constant proportional to M^4 . When this is translated into the dilatation current conservation law, the leading anomalous term is constant. Thus it does not fit to the idea of the PCDC, which implies that the dilatation current should be conserved if the dilaton is massless, and any violation must be proportional to the dilaton mass term.

Under the generalized scale symmetry, i.e. with eq.(4.1), we indeed have the anomalous dilatation conservation law that meets the idea of the PCDC. From the dilaton effective potential

eq.(4.17), the generalized scale symmetry breaking leads to

$$\partial_\mu S_Q^\mu = -\frac{m_\phi^2}{\kappa}\phi \quad (4.18)$$

for small ϕ , where the dilaton mass is given by

$$m_\phi^2 = \frac{\kappa^2 M^4}{8\pi^2} \exp \left\{ 2 \left(1 - \frac{\pi^2}{\kappa^4 \Lambda} \right) \right\}. \quad (4.19)$$

Note that this dilaton mass is precisely the one arises in $V_{\phi,\text{eff}}$ after shifting the vacuum as $\phi \rightarrow \phi + \langle \phi \rangle$, hence confirming that our notion of spontaneous scale symmetry breaking is consistent. The RHS of eq.(4.19) is not really an anomaly, but takes an analogous role here.

Note that $m_\phi^2 \ll \kappa^2 M^4$ for $\langle \phi \rangle \sim 1/\kappa$. In other words, the dilaton mass can be much smaller than the dilaton scale, giving us an expectation that the effect of scale symmetry breaking can be observed in much lower energy scale. This is much different from the Higgs case, where the Higgs mass is comparable to or even heavier than the electroweak symmetry breaking scale.

If the dilaton scale is the same as chiral symmetry breaking scale, say, 100 MeV, then the dilaton could be as light as a few KeV. The dilaton could be a candidate for dark matter. The dilaton couples to fermions and scalars quite universally except the gauge fields, it could be abundant everywhere in the universe. Thus it is worth while to further investigate physics of low energy dilaton, despite that there are not many scalar particles observed in the low energy region.

Even if the low energy dilaton does not exist, it still does not mean the dilaton may cause a phenomenological disaster. The dilaton scale could be very high, making the dilaton massive enough to decay into other particles rapidly at high energy.

5. Conclusions

There have been investigations to understand the relation between the scale symmetry in flat spacetime and the Weyl invariance in curved spacetime without explicitly introducing the dilaton[15][28] and with the dilaton as Brans-Dicke field[13]. In our case, we have shown that the scale symmetry in flat space can be more elegantly described by the Diff symmetry of curved spacetime. The dilaton is correctly identified only if SDiff is incorporated. Being non-scalar, the dilaton is not the Brans-Dicke field. Since scale invariant phenomenological Lagrangians can be derived naturally, we believe this is the structure relevant to low energy dilatation physics.

The dilatation current is derived and we have shown that there is an additional term due to the dilaton, compared to the case without the dilaton. Conformal invariance is a natural consequence of scale invariance. This also proves that two dimensional Liouville theory with the exponential potential term is indeed a conformal field theory.

In this paper we focussed mainly on the implications of spontaneous scale symmetry breaking in four dimensions. But we expect the same idea could be accommodated in studying the quantum Liouville theory.

One subtlety we have not fully explained is if the 1PI effective potential we derived is really unambiguously defined. The reason we believe so is that the effective field theory is assumed to be the low energy realization of a finite or renormalizable theory. Although there are infinite orders of interactions involved, a proper structure of counter terms might exist to make the computation reasonable. Perhaps this could be checked more precisely by investigating any cohomological structure of counter terms[29].

To break scale symmetry involving a dilaton vacuum expectation value is an important task to accomplish in the context of string theory or supergravity. We hope a supersymmetric generalization of the structure presented here would shed new light on that. The best way to supersymmetrize might be to use a four dimensional analog to the super-Liouville theory.

There are also much more works needed to realize the low energy dilaton in nature. Just to name a few directions: The dilaton could be a source of dark matter. There might be interesting dilaton-axion dynamics. Is there any relation between scale symmetry breaking and chiral symmetry breaking? Any relevance in two-Higgs-doublet electroweak symmetry breaking? ...etc. In particular, if the dilaton scale is the same as chiral symmetry breaking scale, then one could think about generalizing chiral Lagrangians to incorporate the dilaton. The simplest way is to dress $\Sigma = \exp(i\pi_a t^a)$ so that $\Sigma_\chi \equiv \chi\Sigma$, which naturally reproduces the linear sigma model Lagrangian. In the usual Goldstone-boson dynamics based on the linear sigma model such a contribution of χ is often neglected in the energy scale lower than the symmetry breaking scale. We now have a new motivation to look into this more carefully.

We hope the results obtained here would be helpful for the future progress on dilatations. Further results will be presented soon.

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